

# SPACES OF IDEMPOTENT MEASURES OF COMPACT METRIC SPACES

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**ABSTRACT.** We investigate certain geometric properties of the spaces of idempotent measures. In particular, we prove that the space of idempotent measures on an infinite compact metric space is homeomorphic to the Hilbert cube.

## 1. INTRODUCTION

The functor  $P$  of probability measures which acts on the category **Comp** of compact metrizable spaces has been investigated by many authors (see e.g., the survey [6]). Geometric properties of spaces of the form  $P(X)$  were established, e.g., in [5]. In particular, it was proved in [5] that the map  $P(f): P(X) \rightarrow P(Y)$  is a trivial  $Q$ -bundle (i.e., a trivial bundle whose fiber is the Hilbert cube  $Q$ ) for an open map of finite-dimensional compact metric spaces with infinite fibers.

The space of idempotent probability measures was systematically studied in [16] (see also [18]), where it was proved, in particular, that the space  $I(X)$  of idempotent probability measures on a topological space  $X$  is compact Hausdorff if such is also  $X$ . The aim of this paper, which can be considered as a continuation of [16], is to establish certain geometric properties of the functor  $I$ .

In particular, we shall prove that  $I(X)$  is homeomorphic to the Hilbert cube for every infinite compact metric space  $X$ . The construction of idempotent measures is functorial in the category of compact Hausdorff spaces and we consider also the geometry of the maps  $I(f)$ , for some maps  $f$ . In particular, we show that, like in the case of probability measures, there exists an open map  $f: X \rightarrow Y$  of compact metric spaces such that  $f$  has infinite fibers and the map  $I(f)$  is not a trivial  $Q$ -bundle.

The paper is organized as follows. In Section 3 we provide the necessary information concerning the spaces of idempotent measures. Section 4 is devoted to (pseudo)metrization of the spaces  $I(X)$ , for a metric space  $X$ . The main results on the topology of the spaces  $I(X)$ , for compact metric spaces  $X$ , are given in Section 5. We also consider the geometry of the maps  $I(f)$ , for some maps  $f$  of compact metric spaces and this allows us to describe the topology of the spaces  $I(X)$  for some nonmetrizable compact Hausdorff spaces  $X$  (Section 6).

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*Date:* March 29, 2008.

*2000 Mathematics Subject Classification.* Primary 54C65, 52A30; Secondary 28A33.

*Key words and phrases.* Compact metric space, Hilbert cube bundle, idempotent measure, functor  $P$  of probability measures, Maslov integral, absolute retract, disjoint approximation property, continuous selection, multivalued map .

## 2. PRELIMINARIES

The space  $Q = \prod_{i=1}^{\infty} [0, 1]_i$  is called the *Hilbert cube*. Recall that an *absolute retract* (AR) is a metrizable space which is a retract of every space in which it lies as a closed subset. The following characterization theorem was proved in [13].

**Theorem 2.1** (Toruńczyk's characterization theorem). *A compact metric space  $X$  is homeomorphic to the Hilbert cube if the following two conditions are satisfied:*

- (1)  *$X$  is an absolute retract;*
- (2)  *$X$  satisfies the disjoint approximation property (DAP), i.e. every two maps of a metric space into  $X$  can be approximated by maps with disjoint images.*

The following notion was introduced in [7]. A *c-structure* on a topological space  $X$  is an assignment to every nonempty finite subset  $A$  of  $X$  a contractible subspace  $F(A)$  of  $X$  such that  $F(A) \subset F(A')$  whenever  $A \subset A'$ . A pair  $(X, F)$ , where  $F$  is a *c-structure* on  $X$  is called a *c-space*. A subset  $E$  of  $X$  is called an *F-set* if  $F(A) \subset E$  for any finite  $A \subset E$ . A metric space  $(X, d)$  is said to be a *metric l.c.-space* if all the open balls are *F-sets* and all open  $r$ -neighborhoods of *F-sets* are also *F-sets*. In fact, it was proved in [8] that every compact metric *l.c.-space* is an AR.

A map  $f: X \rightarrow Y$  is *trivial Q-bundle* if  $f$  is homeomorphic to the projection map  $p_1: Y \times Q \rightarrow Y$ . The following definition is due to Shchepin [11].

**Definition 2.2.** A map  $f: X \rightarrow Y$  is said to be *soft* provided that for every commutative diagram

$$(1) \quad \begin{array}{ccc} A & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow f \\ Z & \xrightarrow{\psi} & Y \end{array}$$

such that  $Z$  is a paracompact space and  $A$  is a closed subset of  $Z$  there exists a map  $\Phi: Z \rightarrow X$  such that  $f\Phi = \psi$  and  $\Phi|_A = \varphi$ .

A map  $f: X \rightarrow Y$  of compact metric spaces is said to satisfy the *fibrewise disjoint approximation property* if, for every  $\varepsilon > 0$ , there exist maps  $g_1, g_2: X \rightarrow Y$  such that

- (1)  $fg_1 = fg_2 = f$ ;
- (2)  $d(1_X, g_i) < \varepsilon$ ,  $i = 1, 2$ ;
- (3)  $g_1(X) \cap g_2(X) = \emptyset$ .

The following result was proved in [14].

**Theorem 2.3** (Toruńczyk-West characterization theorem for  $Q$ -manifold bundles). *A map  $f: X \rightarrow Y$  of compact metric ANR-spaces is a trivial  $Q$ -bundle if  $f$  is soft and  $f$  satisfies the fibrewise disjoint approximation property.*

The following is a generalization of the Michael Selection Theorem – see [8] for the proof. Recall that a multivalued map  $F: X \rightarrow Y$  of topological spaces is called *lower semicontinuous* if, for any open subset  $U$  of  $Y$ , the set  $\{x \in X \mid F(x) \cap U \neq \emptyset\}$  is open in  $X$ . A *selection* of a multivalued map  $F: X \rightarrow Y$  is a (single-valued) map  $f: X \rightarrow Y$  such that  $f(x) \in F(x)$ , for every  $x \in X$ .

**Theorem 2.4.** *Let  $(X, d, F)$  be a complete metric l.c.-space. Then any lower semi-continuous multivalued map  $T: Y \rightarrow X$  of a paracompact space  $Y$  whose values are nonempty closed  $F$ -sets has a continuous selection.*

### 3. SPACES OF IDEMPOTENT MEASURES

In the sequel, all maps will be assumed to be continuous. Let  $X$  be a compact Hausdorff space. We shall denote the Banach space of continuous functions on  $X$  endowed with the sup-norm by  $C(X)$ . For any  $c \in \mathbb{R}$  we shall denote the constant function on  $X$  taking the value  $c$  by  $c_X$ . We shall denote the weight of a topological space  $X$  by  $w(X)$ .

Let  $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$  be endowed with the metric  $\varrho$  defined by  $\varrho(x, y) = |e^x - e^y|$ . Let also  $\mathbb{R}_{\max}^n = (\mathbb{R}_{\max})^n$ . Following the notation of idempotent mathematics (see e.g., [9]) we shall denote by  $\odot: \mathbb{R} \times C(X) \rightarrow C(X)$  the map acting by  $(\lambda, \varphi) \mapsto \lambda_X + \varphi$ , and by  $\oplus: C(X) \times C(X) \rightarrow C(X)$  the map acting by  $(\varphi, \psi) \mapsto \max\{\varphi, \psi\}$ . For each  $c \in \mathbb{R}$  by  $c_X$  we shall denote the constant function from  $C(X)$  defined by the formula  $c_X(x) = c$  for each  $x \in X$ .

**Definition 3.1.** A functional  $\mu: C(X) \rightarrow \mathbb{R}$  is called an *idempotent probability measure* (a *Maslov measure*) if

- (1)  $\mu(c_X) = c$ ;
- (2)  $\mu(c \odot \varphi) = c \odot \mu(\varphi)$ ;
- (3)  $\mu(\varphi \oplus \psi) = \mu(\varphi) \oplus \mu(\psi)$ ,

for every  $\varphi, \psi \in C(X)$ .

The number  $\mu(\varphi)$  is the *Maslov integral* of  $\varphi \in C(X)$  with respect to  $\mu$ . Let  $I(X)$  denote the set of all idempotent probability measures on  $X$ . We endow  $I(X)$  with the weak\* topology. A basis of this topology is formed by the sets

$$\langle \mu; \varphi_1, \dots, \varphi_n; \varepsilon \rangle = \{ \nu \in I(X) \mid |\mu(\varphi_i) - \nu(\varphi_i)| < \varepsilon, i = 1, \dots, n \},$$

where  $\mu \in I(X)$ ,  $\varphi_i \in C(X)$ ,  $i = 1, \dots, n$ , and  $\varepsilon > 0$ .

The following is an example of an idempotent probability measure. Let  $x_1, \dots, x_n \in X$  and  $\lambda_1, \dots, \lambda_n \in \mathbb{R}_{\max}$  be numbers such that  $\max\{\lambda_1, \dots, \lambda_n\} = 0$ . Define  $\mu: C(X) \rightarrow \mathbb{R}$  as follows:  $\mu(\varphi) = \max\{\varphi(x_i) + \lambda_i \mid i = 1, \dots, n\}$ . As usual, for every  $x \in X$ , we denote by  $\delta_x$  (or  $\delta(x)$ ) the functional on  $C(X)$  defined as follows:  $\delta_x(\varphi) = \varphi(x)$ ,  $\varphi \in C(X)$  (the Dirac probability measure concentrated at  $x$ ). Then one can write  $\mu = \oplus_{i=1}^n \lambda_i \odot \delta_{x_i}$ .

Given a map  $f: X \rightarrow Y$  of compact Hausdorff spaces, the map  $I(f): I(X) \rightarrow I(Y)$  is defined by the formula  $I(f)(\mu)(\varphi) = \mu(\varphi f)$ , for every  $\varphi \in C(Y)$ . That  $I(f)$  is continuous and that  $I$  is a covariant functor acting in the category **Comp** is proved in [16]. Note that, if  $\mu = \oplus_{i=1}^n \lambda_i \odot \delta_{x_i} \in I(X)$ , then  $I(f)(\mu) = \oplus_{i=1}^n \lambda_i \odot \delta_{f(x_i)} \in I(Y)$ .

**3.1. Milyutin maps.** The following result was proved in [16].

**Theorem 3.2.** *Let  $X$  be a compact metrizable space. Then there exists a zero-dimensional compact metrizable space  $Y$  and a continuous map  $f: X \rightarrow Y$  for which there exists a continuous map  $s: Y \rightarrow I(X)$  such that  $\text{supp}(y) \subset f^{-1}(y)$ , for every  $y \in Y$ .*

A map  $f$  satisfying the conditions stated above is called a *Milyutin map* of idempotent measures.

**3.2. Map  $\zeta_X$ .** Given  $M \in I^2(X)$ , define the map  $\zeta_X(M): C(X) \rightarrow \mathbb{R}$  as follows:  $\zeta_X(M)(\varphi) = M(\bar{\varphi})$ . Given  $\varphi \in C(X)$ , define  $\bar{\varphi}: I(X) \rightarrow \mathbb{R}$  as follows:  $\bar{\varphi}(\mu) = \mu(\varphi)$ ,  $\mu \in I(X)$ . It is proved in [16] that the map  $\zeta_X$  is continuous.

#### 4. METRIZATION

Let  $(X, d)$  be a compact metric space. By  $n - \text{LIP} = n - \text{LIP}(X, d)$  we denote the set of Lipschitz functions with the Lipschitz constant  $\leq n$  from  $C(X)$ . Fix  $n \in \mathbb{N}$ . For every  $\mu, \nu$ , let

$$\hat{d}_n(\mu, \nu) = \sup\{|\mu(\varphi) - \nu(\varphi)| \mid \varphi \in n - \text{LIP}\}.$$

**Theorem 4.1.** *The function  $\hat{d}_n$  is a continuous pseudometric on  $I(X)$ .*

*Proof.* We first remark that  $\hat{d}_n$  is well-defined. Indeed,  $\sup \varphi - \inf \varphi \leq n \text{diam } X$ , for every  $\varphi \in n - \text{LIP}$ , whence  $|\mu(\varphi) - \nu(\varphi)| \leq 2n \text{diam } X$ .

Obviously,  $\hat{d}_n(\mu, \mu) = 0$  and  $\hat{d}_n(\mu, \nu) = \hat{d}_n(\nu, \mu)$ , for every  $\mu, \nu \in I(X)$ .

We are going to prove that  $\hat{d}$  satisfies the triangle inequality. Since, for every  $\varphi \in n - \text{LIP}$  and  $\mu, \nu, \tau \in I(X)$ ,

$$\hat{d}(\mu, \nu) \geq |\mu(\varphi) - \nu(\varphi)|, \quad \hat{d}(\nu, \tau) \geq |\nu(\varphi) - \tau(\varphi)|,$$

we have

$$\hat{d}_n(\mu, \nu) + \hat{d}_n(\nu, \tau) \geq |\mu(\varphi) - \nu(\varphi)| + |\nu(\varphi) - \tau(\varphi)| \geq |\mu(\varphi) - \tau(\varphi)|,$$

whence, passing to sup in the right-hand side, we obtain  $\hat{d}_n(\mu, \nu) + \hat{d}_n(\nu, \tau) \geq \hat{d}_n(\mu, \tau)$ .

Now, we prove that  $\hat{d}$  is continuous. Suppose to the contrary. Then one can find a sequence  $(\mu_i)_{i=1}^\infty$  in  $I(X)$  such that  $\lim_{i \rightarrow \infty} \mu_i = \mu \in I(X)$  and  $\hat{d}(\mu_i, \mu) \geq c'$ , for some  $c' > 0$ . Then there exist  $\varphi_i \in n - \text{LIP}$ ,  $i \in \mathbb{N}$ , such that  $|\mu_i(\varphi_i) - \mu(\varphi_i)| \geq c$ , for some  $c > 0$ . Since the functionals in  $I(X)$  are weakly additive, without loss of generality, one may assume that  $\varphi_i(x_0) = 0$ , for some base point  $x_0 \in X$ ,  $i \in \mathbb{N}$ . By the Arzela-Ascoli theorem, there exists a limit point  $\varphi \in n - \text{LIP}$  of the sequence  $(\varphi_i)_{i=1}^\infty$ . We have  $|\mu_i(\varphi) - \mu(\varphi)| \geq c$ , which contradicts to the fact that  $(\mu_i)_{i=1}^\infty$  converges to  $\mu$ .  $\square$

**Remark 4.2.** Simple examples demonstrate that  $\hat{d}$  cannot be a metric whenever  $X$  consists of more than one point.

**Proposition 4.3.** *The family of pseudometrics  $\hat{d}_n$ ,  $n \in \mathbb{N}$ , separates the points in  $I(X)$ .*

*Proof.* Let  $\mu, \nu \in I(X)$ ,  $\mu \neq \nu$ . There exists  $\varphi \in C(X)$  such that  $|\mu(\varphi) - \nu(\varphi)| > c$ , for some  $c > 0$ . There exists  $\psi \in n - \text{LIP}$ , for some  $n \in \mathbb{N}$ , such that  $\|\varphi - \psi\| \leq (c/3)$ . Then, clearly,  $|\mu(\psi) - \nu(\psi)| \geq (c/3)$  and therefore  $\hat{d}_n(\mu, \nu) \geq (c/3)$ .  $\square$

We let  $\tilde{d}_n = (1/n)\hat{d}_n$ .

**Proposition 4.4.** *The map  $\delta = \delta_X$ ,  $x \mapsto \delta_x: (X, d) \rightarrow (I(X), \tilde{d}_n)$ , is an isometric embedding for every  $n \in \mathbb{N}$ .*

*Proof.* Let  $x, y \in X$  and  $\varphi \in \mathfrak{n} - \text{LIP}$ . Then  $|\delta_x(\varphi) - \delta_y(\varphi)| \leq nd(x, y)$ , therefore  $\hat{d}_n(\delta_x, \delta_y) \leq nd(x, y)$ . Thus  $\tilde{d}_n(\delta_x, \delta_y) \leq d(x, y)$ .

On the other hand, define  $\varphi_x \in \mathfrak{n} - \text{LIP}$  by the formula  $\varphi_x(z) = nd(x, z)$ ,  $z \in X$ . Then  $|\delta_x(\varphi_x) - \delta_y(\varphi_x)| = nd(x, y)$  and we are done.  $\square$

**Proposition 4.5.** *Let  $f: (X, d) \rightarrow (Y, \varrho)$  be a nonexpanding map of compact metric spaces. Then the map  $I(f): (I(X), \hat{d}_n) \rightarrow (I(Y), \hat{\varrho}_n)$  is also nonexpanding, for every  $n \in \mathbb{N}$ .*

*Proof.* Given  $\varphi \in \mathfrak{n} - \text{LIP}(Y)$ , note that  $\varphi f \in \mathfrak{n} - \text{LIP}(X)$  and, for any  $\mu, \nu \in I(X)$ , we have

$$|I(f)(\mu)(\varphi) - I(f)(\nu)(\varphi)| = |\mu(\varphi f) - \nu(\varphi f)| \leq \hat{d}_n(\mu, \nu).$$

Passing to the limit in the left-hand side of the above formula, we are done.  $\square$

Note that the above construction of  $\hat{d}$  can be applied not only to metrics but also to continuous pseudometrics. Proceeding in this way we obtain the iterations  $(I(X), \tilde{d}_n)$ ,  $(I^2(X), \tilde{\tilde{d}}_{nm} = (\tilde{d}_n \tilde{\sim}_m), \dots$

**Proposition 4.6.** *For a metric space  $(X, d)$ , the map  $\zeta_X: (I^2(X), \tilde{\tilde{d}}_{nn}) \rightarrow (I(X), \tilde{d}_n)$  is nonexpanding.*

*Proof.* We first prove that, for any  $\varphi \in \mathfrak{n} - \text{LIP}(X, d)$ , we have  $\bar{\varphi} \in \mathfrak{n} - \text{LIP}(I(X), \hat{d})$ . Indeed, given  $\mu, \nu \in I(X)$ , we see that

$$n\tilde{d}(\mu, \nu) = \hat{d}(\mu, \nu) \geq |\mu(\varphi) - \nu(\varphi)| = |\bar{\varphi}(\mu) - \bar{\varphi}(\nu)|$$

and we are done.

Suppose now that  $M, N \in I^2(X)$ ,  $\mu = \zeta_X(M)$ ,  $\nu = \zeta_X(N)$ . Given  $\varphi \in \mathfrak{n} - \text{LIP}(X, d)$ , we obtain

$$|\mu(\varphi) - \nu(\varphi)| = |M(\bar{\varphi}) - N(\bar{\varphi})| \leq \tilde{\tilde{d}}_{nn}(M, N).$$

Passing to the limit in the left-hand side, we are done.  $\square$

**Remark 4.7.** Using the results on existence of the pseudometrics  $\tilde{d}_n$ , one can define the spaces of idempotent probability measures with compact support for metric and, more generally, uniform spaces. Indeed, let  $(X, d)$  be a metric space. We define the set  $I(X)$  to be the direct limit of the direct system  $\{I(A), I(\iota_{AB}); \exp X\}$  (here, for  $A, B \in \exp X$  with  $A \subset B$ , we denote by  $\iota_{AB}: A \rightarrow B$  the inclusion map). For every  $A \in \exp X$ , we identify  $I(A)$  with the corresponding subset of  $I(X)$  along the map  $I(\iota_A)$ , where  $\iota_A: A \rightarrow X$  is the limit inclusion map. For any  $\mu \in I(X)$ , there exists a unique minimal  $A \in \exp X$  such that  $\mu \in I(A)$ . Then we say that  $A$  is the *support* of  $\mu$  and write  $\text{supp}(\mu) = A$ .

Now, define a family of pseudometrics  $\hat{d}_n$ ,  $n \in \mathbb{N}$ , on  $I(X)$  as follows. Given  $\mu, \nu \in I(X)$ , we let

$$\hat{d}_n(\mu, \nu) = \hat{d}_n|((\text{supp}(\mu) \cup \text{supp}(\nu)) \times (\text{supp}(\mu) \cup \text{supp}(\nu)))(\mu, \nu).$$

One can prove that, for any uniform space  $(X, \mathcal{U})$ , if the uniformity  $\mathcal{U}$  is generated by a family  $\{d^\alpha \mid \alpha \in A\}$  of pseudometrics, then the family  $\{\tilde{d}_n^\alpha \mid \alpha \in A, n \in \mathbb{N}\}$  of pseudometrics on  $I(X)$  generates a uniformity on  $I(X)$ .

Let  $(X, d)$  be a compact metric space. We define a function  $\tilde{d}: I(X) \times I(X) \rightarrow \mathbb{R}$  as follows:

$$\tilde{d}(\mu, \nu) = \sum_{i=1}^{\infty} \frac{\tilde{d}_n(\mu, \nu)}{2^i}.$$

It follows from what was proved above that  $\tilde{d}$  is a metric on the space  $I(X)$ .

## 5. SPACE OF IDEMPOTENT MEASURES FOR METRIC COMPACTA

It is proved in [16] that the set  $I(X)$  is homeomorphic to the  $(n-1)$ -dimensional simplex for any finite  $X$  with  $|X| = n$ .

A set  $A \subset I(X)$  is called *max-plus convex* if, for every  $\mu, \nu \in A$  and every  $\alpha, \beta \in \mathbb{R}_{\max}$  with  $\alpha \oplus \beta = 0$ , we have  $\alpha \odot \mu \oplus \beta \odot \nu \in A$ .

**Lemma 5.1.** *Let  $\mu_0 \in I(X)$ . The map  $h: I(X) \times [-\infty, 0] \rightarrow I(X)$ ,  $h(\mu, \lambda) = \mu \oplus (\lambda \odot \mu_0)$ , is continuous.*

*Proof.* Let  $(\mu, \lambda) \in I(X) \times [-\infty, 0]$ ,  $\nu = h(\mu, \lambda)$ , and  $\langle \nu; \varphi; \varepsilon \rangle$  be a subbase neighborhood of  $\nu$ .

Case 1).  $h(\mu, \lambda) = \mu(\varphi)$ . Then  $\mu(\varphi) \geq \lambda + \mu_0(\varphi)$  and it is evident that, for any  $\mu' \in \langle \mu; \varphi; \varepsilon \rangle$  and  $\lambda' \in [-\infty, \lambda + \varepsilon] \cap \mathbb{R}_{\max}$ , we have  $h(\mu', \lambda') \in \langle \nu; \varphi; \varepsilon \rangle$ .

Case 2).  $h(\mu, \lambda) = \lambda + \mu_0(\varphi)$ . Then necessarily  $\lambda > -\infty$ . For every  $\mu' \in \langle \mu; \varphi; \varepsilon \rangle$  and  $\lambda' \in (\lambda + \varepsilon, \lambda + \varepsilon) \cap \mathbb{R}_{\max}$ , we have  $h(\mu', \lambda') \in \langle \nu; \varphi; \varepsilon \rangle$ .  $\square$

**Lemma 5.2.** *Let  $X$  be a compact metrizable space. Every max-plus convex subset in  $I(X)$  is contractible.*

*Proof.* Let  $A \subset I(X)$  be a nonempty max-plus convex subset in  $I(X)$ . Note that  $\max A \in A$ , because of max-plus convexity of  $A$ . Define the map  $H: A \times \mathbb{R}_{\max} \rightarrow A$  as follows:  $H(\mu, \lambda) = \mu \oplus (\lambda \odot \max A)$ . Then  $H(\mu, -\infty) = \mu$  and  $H(\mu, 0) = \max A$ . This shows that the set  $A$  is contractible.  $\square$

**Theorem 5.3.** *The space  $I(X)$  is homeomorphic to the Hilbert cube for any infinite compact metrizable space  $X$ .*

*Proof.* We first show that  $I(X)$  is an AR-space. Fix a metric  $d$  on  $X$  that generates its topology. Define a  $c$ -structure on the space  $I(X)$  as follows. To every nonempty finite subset  $A = \{\mu_1, \dots, \mu_n\}$  of  $I(X)$  assign a subspace

$$F(A) = \{\oplus_{i=1}^n \alpha_i \odot \mu_i \mid \alpha_1, \dots, \alpha_n \in \mathbb{R}_{\max}, \oplus_{i=1}^n \alpha_i = 0\}.$$

It is easy to verify that  $F(A)$  is max-plus convex and therefore contractible.

We are going to show that this  $c$ -structure is an  $l.c.$ -structure. We shall prove that every ball with respect to the metric  $\hat{d}$  in  $I(X)$  is convex.

Let  $\mu, \nu, \tau \in I(X)$ ,  $\lambda \in \mathbb{R}_{\max}$ , and  $\varphi \in \mathfrak{n} - \text{LIP}$ . Then

$$|\mu(\varphi) - ((\lambda \odot \nu) \oplus \tau)(\varphi)| \leq \max\{|\mu(\varphi) - \nu(\varphi)|, |\mu(\varphi) - \tau(\varphi)|\},$$

whence

$$\hat{d}_n(\mu, (\lambda \odot \nu) \oplus \tau) \leq \max\{\hat{d}_n(\mu, \nu), \hat{d}_n(\mu, \tau)\}$$

and therefore, for any  $\varepsilon > 0$ , if  $\hat{d}(\mu, \nu) < \varepsilon$  and  $\hat{d}(\mu, \tau) < \varepsilon$ , then  $\hat{d}(\mu, (\lambda \odot \nu) \oplus \tau) < \varepsilon$ .

We now show that the space  $I(X)$  satisfies DAP. Let  $f: Y \rightarrow X$  be a Milyutin map of a zero-dimensional compact metrizable space  $Y$ . There exists a continuous map  $s: X \rightarrow I(Y)$  such that  $I(f)s(x) = \delta_x$ , for every  $x \in X$ . Let  $r: Y \rightarrow Y'$  be a retraction of  $Y$  onto its finite subset  $Y'$ . Since  $Y$  is zero-dimensional, we may choose  $r$  as close to  $\text{id}_Y$  as we wish. Define  $g_1: I(X) \rightarrow I(X)$  as follows:

$$g_1(\mu) = I(fr)\zeta_Y I(s)(\mu) = \zeta_Y I^2(fr)I(s)(\mu).$$

Then  $g_1(I(X)) \subset I_\omega(X)$ .

Recall that  $j_X(X) \in I(X)$  acts as follows:  $j_X(X)(\varphi) = \max \varphi$ ,  $\varphi \in C(X)$ . Define  $g_2: I(X) \rightarrow I(X)$  by the formula  $g_2(\mu) = \mu \oplus \lambda \odot j_X(X)$ , where  $\lambda \in (-\infty, 0]$  is fixed. If  $\lambda$  is small enough, the map  $g_2$  is close to  $\text{id}_{I(X)}$ .

Note that  $\text{supp}(g_2(\mu)) = X$ , for every  $\mu \in I(X)$ , and therefore  $g_1(I(X)) \cap g_2(I(X)) = \emptyset$ . By Toruńczyk's theorem,  $I(X)$  is homeomorphic to  $Q$ .  $\square$

**Remark 5.4.** Note that, for max-plus convex sets, a version of the Michael Selection Theorem is proved in [17].

## 6. MAPS OF SPACES OF IDEMPOTENT MEASURES

**Theorem 6.1.** *Let  $p_1: X \times Y \rightarrow X$  denote the projection onto the first factor, where  $X, Y$  are compact metric spaces. Then the map  $I(p_1): I(X \times Y) \rightarrow I(X)$  is soft.*

*Proof.* It is proved in [16] that the map  $I(f)$  is open. Given a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & I(X \times Y) \\ \downarrow & & \downarrow I(p_1) \\ Z & \xrightarrow[\psi]{} & I(X) \end{array}$$

$\square$

**Example 6.2.** The following example demonstrates that, like in the case of probability measures, there exists an open map  $f: X \rightarrow Y$  of metrizable compacta with infinite fibers such that the map  $I(f): I(X) \rightarrow I(Y)$  is not a trivial  $Q$ -bundle.

We exploit the construction from [3]. Let  $S^n$  denote the  $n$ -dimensional sphere and  $\mathbb{R}P^n$  the  $n$ -dimensional real projective space. Let  $\eta_n: S^n \rightarrow \mathbb{R}P^n$  denote the canonical map. The required map is

$$f = \prod_{i=1}^{\infty} \eta_{2^i-1}: X = \prod_{i=1}^{\infty} S^{2^i-1} \rightarrow Y = \prod_{i=1}^{\infty} \mathbb{R}P^{2^i-1}.$$

It is proved in [3] that the map

$$f_k = \prod_{i=1}^k \eta_{2^i-1}: \prod_{i=1}^k S^{2^i-1} \rightarrow \prod_{i=1}^k \mathbb{R}P^{2^i-1}$$

has the following property: the map  $P_0(f_k): P_0(\prod_{i=1}^k S^{2^i-1}) \rightarrow \prod_{i=1}^k \mathbb{R}P^{2^i-1}$ , where

$$P_0\left(\prod_{i=1}^k S^{2^i-1}\right) = (P(f_k))^{-1}\left(\left\{\delta_x \mid x \in \prod_{i=1}^k \mathbb{R}P^{2^i-1}\right\}\right)$$

and  $P_0(f_k)$  sends every  $\mu \in P_0\left(\prod_{i=1}^k S^{2^i-1}\right)$  to the unique  $x \in \prod_{i=1}^k \mathbb{R}P^{2^i-1}$  for which  $\text{supp}(\mu) \in f_k^{-1}(x)$ , has no two disjoint selections.

Proceeding similarly as in [3] we reduce the problem of existence of two disjoint sections of the map  $I(f)$  to that of existence of two disjoint selections of the map  $I_0(f_k)$ , for some  $k$ , where the map  $I_0(f_k)$  is defined similarly as  $P_0(f_k)$  with  $P$  replaced by  $I$ .

We only have to show that the maps  $I_0(f_k)$  and  $P_0(f_k)$  are homeomorphic. Let  $\mu = \oplus_{i=1}^n \lambda_i \odot \delta_{x_i} \in P_0(f)$ . Then

$$(e^{\lambda_1}, \dots, e^{\lambda_n}) \in \Gamma^n = \{(z_1, \dots, z_n) \in [0, 1]^n \mid z_1 \oplus \dots \oplus z_n = 1\}.$$

Let

$$\Sigma^{n-1} = \{(z_1, \dots, z_n) \in \Gamma^n \mid z_i = 0 \text{ for some } i\},$$

then there exists a point  $(\Lambda_1, \dots, \Lambda_n) \in \Sigma^{n-1}$  such that

$$(e^{\lambda_1}, \dots, e^{\lambda_n}) = s(\Lambda_1, \dots, \Lambda_n) + (1-s)(1, \dots, 1),$$

for some  $s \in [0, 1]$  (note that such a point  $(\Lambda_1, \dots, \Lambda_n)$  is uniquely determined whenever  $(e^{\lambda_1}, \dots, e^{\lambda_n}) \neq (1, \dots, 1)$ ). We then let

$$(\lambda'_1, \dots, \lambda'_n) = s(\Lambda'_1, \dots, \Lambda'_n) + (1-s)(1/n, \dots, 1/n),$$

where  $(\Lambda'_1, \dots, \Lambda'_n)$  is the point of intersection of the linear segment

$$[(\Lambda_1, \dots, \Lambda_n), (1, \dots, 1)]$$

with the boundary

$$\partial\Delta^{n-1} = \{(x_1, \dots, x_n) \mid \sum_{i=1}^n x_i = 1\}.$$

One can easily verify that  $\mu \mapsto \mu' = \sum_{i=1}^n \lambda'_i \delta_{x_i} \in P_0(f)$  is a desired homeomorphism.

## 7. NONMETRIZABLE CASE

The notion of normal functor in the category **Comp** of compact Hausdorff spaces was introduced in [11].

**Theorem 7.1.** *Let  $\tau > \omega_1$ . Then the set  $I([0, 1]^\tau)$  is not an AR.*

*Proof.* In [11], Shchepin proved that, for any normal functor  $F$  which is not a power functor, any cardinal number  $\tau > \omega_1$ , and any compact metric space  $K$  with  $|K| \geq 2$ , the functor-power  $F(K^\tau)$  is not an AR. In [16], it is proved that  $I$  is a normal functor, whence the result follows.  $\square$

**Theorem 7.2.** *Let  $X$  be an openly generated character-homogeneous compact Hausdorff space of weight  $\omega_1$ . Then the space  $I(X)$  is homeomorphic to  $I^{\omega_1}$ .*

*Proof.* We can represent  $X$  as  $\varprojlim \mathcal{S}$ , where  $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}; \omega_1\}$  is an inverse system such that  $p_{\alpha\beta}: X_\alpha \rightarrow X_\beta$  are open maps for all  $\alpha, \beta < \omega_1$ ,  $\alpha \geq \beta$ , and  $X_\alpha$ ,  $\alpha < \omega_1$ , are infinite compact metric spaces. Since  $X$  is character-homogeneous, we may additionally assume that the maps  $p_{\alpha\beta}$  do not contain singleton fibers.

Then we have  $I(X) = \varprojlim I(\mathcal{S}) = \{I(X_\alpha), I(p_{\alpha\beta}); \omega_1\}$  (see [16]). By Theorem 6.1, the maps  $I(p_{\alpha\beta})$  are soft. Applying arguments from [10] one can find a cofinal subset  $A$  of  $\omega_1$  such that, for every  $\alpha, \beta \in A$ ,  $\alpha > \beta$ , the map  $I(p_{\alpha\beta})$  satisfies the FDAP. Therefore, by the Toruńczyk-West theorem, the map  $I(p_{\alpha\beta})$  is homeomorphic to the projection  $\pi_1: Q \times Q \rightarrow Q$  onto the first factor. In turn,  $I(X)$  is homeomorphic to  $Q^A \simeq Q^{\omega_1} \simeq I^{\omega_1}$ . □

## 8. EPILOGUE

One can also consider the spaces  $I(K^\tau)$ , for arbitrary  $\tau$  and nondegenerate compact metrizable space  $K$ . The interesting results on autohomeomorphisms of the spaces  $P(K^\tau)$ , for  $\tau > \omega_1$ , were obtained by Smurov [12]. One can conjecture that these results have their counterparts also in the case of spaces of idempotent measures.

In connection with the mentioned in the introduction result by Fedorchuk, the following question arises: *Is the map  $I(f): I(X) \rightarrow I(Y)$  is a trivial  $Q$ -bundle, for an open map of finite-dimensional compact metric spaces with infinite fibers?*

As it was remarked above, one can also consider the spaces  $I(X)$  for noncompact metric space  $X$ . It looks plausible that the results on topology of spaces of probability measures proved in [1] should have their counterparts also for the idempotent measures.

## ACKNOWLEDGEMENTS

This research was supported by SRA grants P1-0292-0101-04 and BI-UA/07-08-001.

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